Some properties on G-evaluation and its applications to G-martingale decomposition

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Abstract

In this article, a sublinear expectation induced by G-expectation is introduced, which is called G-evaluation for convenience. As an application, we prove that for any $\xi \in L_G^{\beta}(\Omega_T)$ with some $\beta > 1$ the decomposition theorem holds and that any $\beta > 1$ integrable symmetric G-martingale can be represented as an Itô integral w.r.t G-Brownian motion. As a byproduct, we prove a regular property for G-martingale: Any G-martingale $\{M_t\}$ has a quasi-continuous version.

Key words: G-expectation, G-evaluation, G-martingale, Decomposition theorem

MSC-classification: 60G07, 60G20, 60G44, 60G48, 60H05

1 Introduction

Recently, [P06], [P08] introduced the notion of sublinear expectation space, which is a generalization of probability space. One of the most important sublinear expectation space is G-expectation space. As the counterpart of Wiener space in the linear case, the notions of G-Brownian motion, G-martingale, and Itô integral w.r.t G-Brownian motion were also introduced. These notions have very rich and interesting new structures which nontrivially generalize the classical ones.

Because of the Sublinearity, the fact of $\{M_t\}$ being a G-martingale does not imply that $\{-M_t\}$ is a G-martingale. A surprising fact is that there

exist nontrivial processes which are continuous, decreasing and are also G-martingales. [P07] conjectured that for any $\xi \in L_G^1(\Omega_T)$, we have the following representation:

$$X_t := \hat{E}_t(\xi) = \hat{E}(\xi) + \int_0^t Z_s dB_s - K_t,\ t \in [0,T],$$

with $K_0 = 0$ and $\{K_t\}$ an increasing process.

[P07] proved the conjecture for cylindrical functions $L_{ip}(\Omega_T)$(see Theorem 2.18) by Itô's formula in the setting of G-expectation space. So the left question is to extend the representation to the completion $L_G^1(\Omega_T)$ of $L_{ip}(\Omega_T)$ under norm $\|\xi\|_{1,G} = \hat{E}(|\xi|)$.

[STZ09] make a progress in this direction. They define a much stronger norm $\|\xi\|_{\mathcal{L}_2^0} = \{\hat{E}[\sup_{t\in[0,T]} \hat{E}_t(|\xi|^2)]\}^{1/2}$ on $L_{ip}(\Omega_T)$ and generalized the above result to the completion $\mathcal{L}_2^0 \subset L_G^2(\Omega_T)$. The shortcoming of this result is that no relations between the two norms are given. The space $\mathcal{L}_2^0$ is just an abstract completion and we have no idea about the set $L_G^2(\Omega_T)\backslash\mathcal{L}_2^0$.

The purpose of this article is to improve the decomposition theorem given in [P07] and [STZ09]. The main results of the article consist of:

We introduce a sublinear expectation called G-evaluation and investigate its properties. By presenting an estimate, which can be seen as the substitute of Doob's maximal inequality, we proved that for any $\xi \in L_G^\beta(\Omega_T)$ with some $\beta > 1$ the decomposition theorem holds and that any $\beta > 1$ integrable symmetric G-martingale can be represented as an Itô integral w.r.t G-Brownian motion.

As a byproduct, we prove a regular property for G-martingale: Any G-martingale $\{M_t\}$ has a quasi-continuous version(see Definition 5.1). We also give several estimates for variables in the decomposition theorem, which may be useful in the follow-up work of G-martingale theory.

This article is organized as follows: In section 2, we recall some basic notions and results of G-expectation and the related space of random variables. In section 3, we introduce the notion of G-evaluation and present an estimate, which can be seen as the substitute of Doob's maximal inequality. In section 4, we prove that for any $\beta > 1$ integrable G-martingale, the decomposition theorem holds and that any $\beta > 1$ integrable symmetric G-martingale can be represented as an Itô integral w.r.t G-Brownian motion. In section 5, we prove a regular property for G-martingale.

2 Preliminary

We present some preliminaries in the theory of sublinear expectations and the related G-Brownian motions. More details of this section can be found in [P07].

2.1 G-expectation

Definition 2.1 Let Ω be a given set and let $\mathcal{H}$ be a linear space of real valued functions defined on Ω with $c \in \mathcal{H}$ for all constants c. $\mathcal{H}$ is considered as the space of random variables. A sublinear expectation $\hat{E}$ on $\mathcal{H}$ is a functional $\hat{E} : \mathcal{H} \to R$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

(a) Monotonicity: If $X \geq Y$ then $\hat{E}(X) \geq \hat{E}(Y)$.

(b) Constant preserving: $\hat{E}(c) = c$.

(c) Sub-additivity: $\hat{E}(X) - \hat{E}(Y) \leq \hat{E}(X - Y)$.

(d) Positive homogeneity: $\hat{E}(\lambda X) = \lambda \hat{E}(X)$, $\lambda \geq 0$.

$(\Omega, \mathcal{H}, \hat{E})$ is called a sublinear expectation space.

Definition 2.2 Let X_1 and X_2 be two n-dimensional random vectors defined respectively in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{E}_2)$. They are called identically distributed, denoted by $X_1 \sim X_2$, if $\hat{E}_1[\varphi(X_1)] = \hat{E}_2[\varphi(X_2)]$, $\forall \varphi \in C_{l,Lip}(R^n)$, where $C_{l,Lip}(R^n)$ is the space of real continuous functions defined on R^n such that

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^k + |y|^k)|x - y|, \forall x, y \in R^n,$$

where k depends only on φ.

Definition 2.3 In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ a random vector $Y = (Y_1, \cdot \cdot \cdot, Y_n)$, $Y_i \in \mathcal{H}$ is said to be independent to another random vector $X = (X_1, \cdot \cdot \cdot, X_m)$, $X_i \in \mathcal{H}$ under $\hat{E}(\cdot)$ if for each test function $\varphi \in C_{l,Lip}(R^m \times R^n)$ we have $\hat{E}[\varphi(X, Y)] = \hat{E}[\hat{E}[\varphi(x, Y)]_{x=X}]$.

Definition 2.4 (G-normal distribution) A d-dimensional random vector $X = (X_1, \cdot \cdot \cdot, X_d)$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ is called G-normal distributed if for each $a, b \in R$ we have

$$aX + b\hat{X} \sim \sqrt{a^2 + b^2} X,$$

where $\hat{X}$ is an independent copy of X. Here the letter G denotes the function

$$G(A) := \frac{1}{2} \hat{E}[(AX, X)] : S_d \to R,$$

where S_d denotes the collection of $d \times d$ symmetric matrices.

The function $G(\cdot) : S_d \to R$ is a monotonic, sublinear mapping on S_d and $G(A) = \frac{1}{2}\hat{E}[(AX, X)] \le \frac{1}{2}|A|\hat{E}[|X|^2] =: \frac{1}{2}|A|\bar{\sigma}^2$ implies that there exists a bounded, convex and closed subset $\Gamma \subset S_d^+$ such that

$$G(A) = \frac{1}{2}\sup_{\gamma \in \Gamma} Tr(\gamma A).$$

If there exists some $\beta > 0$ such that $G(A) - G(B) \ge \beta Tr(A - B)$ for any $A \ge B$, we call the G-normal distribution is non-degenerate, which is the case we consider throughout this article.

Definition 2.5 i) Let $\Omega_T = C_0([0,T]; R^d)$ with the supremum norm, $\mathcal{H}_T^0 := \{\varphi(B_{t_1}, ..., B_{t_n}) | \forall n \ge 1, t_1, ..., t_n \in [0,T], \forall \varphi \in C_{l,Lip}(R^{d\times n})\}$, G-expectation is a sublinear expectation defined by

$$\hat{E}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})]$$

$$= \tilde{E}[\varphi(\sqrt{t_1 - t_0}\xi_1, \cdots, \sqrt{t_m - t_{m-1}}\xi_m)],$$

for all $X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})$, where $\xi_1, \cdots, \xi_n$ are identically distributed d-dimensional G-normal distributed random vectors in a sublinear expectation space $(\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{E})$ such that ξ_{i+1} is independent to $(\xi_1, \cdots, \xi_i)$ for each $i = 1, \cdots, m$. $(\Omega_T, \mathcal{H}_T^0, \hat{E})$ is called a G-expectation space.

ii) For $t \in [0,T]$ and $\xi = \varphi(B_{t_1}, ..., B_{t_n}) \in \mathcal{H}_T^0$, the conditional expectation defined by(there is no loss of generality, we assume $t = t_i$)

$$\hat{E}_{t_i}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})]$$

$$= \tilde{\varphi}(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_i} - B_{t_{i-1}}),$$

where

$$\tilde{\varphi}(x_1, \cdots, x_i) = \hat{E}[\varphi(x_1, \cdots, x_i, B_{t_{i+1}} - B_{t_i}, \cdots, B_{t_m} - B_{t_{m-1}})].$$

Let $\|\xi\|_{p,G} = [\hat{E}(|\xi|^p)]^{1/p}$ for $\xi \in \mathcal{H}_T^0$ and $p \ge 1$, then $\forall t \in [0,T]$, $\hat{E}_t(\cdot)$ is a continuous mapping on $\mathcal{H}_T^0$ with norm $\|\cdot\|_{1,G}$ and therefore can be extended continuously to the completion $L_G^1(\Omega_T)$ of $\mathcal{H}_T^0$ under norm $\|\cdot\|_{1,G}$.

Proposition 2.6 Conditional expectation defined above has the following properties: for $X, Y \in L_G^1(\Omega_T)$

i) If $X \ge Y$, then $\hat{E}_t(X) \ge \hat{E}_t(Y)$.

ii) $\hat{E}_t(\eta) = \eta$, for $\eta \in L_G^1(\Omega_t)$.

iii) $\hat{E}_t(X) - \hat{E}_t(Y) \le \hat{E}_t(X - Y)$.

iv) $\hat{E}_t(\eta X) = \eta^+ \hat{E}_t(X) + \eta^- \hat{E}_t(-X)$, for each bounded $\eta \in L^1_G(\Omega_t)$.

v) $\hat{E}_s(\hat{E}_t(X)) = \hat{E}_{t\wedge s}(X)$, in particular, $\hat{E}(\hat{E}_t(X)) = \hat{E}(X)$.

vi) For each $X \in L^1_G(\Omega^t_T)$ we have $\hat{E}_t(X) = \hat{E}(X)$.

Theorem 2.7([DHP08]) There exists a tight subset $\mathcal{P} \subset \mathcal{M}_1(\Omega_T)$ such that

$$\hat{E}(\xi) = \max_{P\in\mathcal{P}} E_P(\xi) \text{ for all } \xi \in \mathcal{H}^0_T.$$

$\mathcal{P}$ is called a set that represents $\hat{E}$.

Remark 2.8 i) [HP09] gave a new proof to the above theorem. From the proof we know that any sublinear expectation $\mathcal{E}(\cdot)$ on $\mathcal{H}^0_T$ satisfying

$$\mathcal{E}[(B_t - B_s)^{2n}] \le d_n(t-s)^n, \forall n \in N,$$

has the above representation.

ii) Let $\mathcal{A}$ denotes the sets that represent $\hat{E}$. $\mathcal{P}^* = \{P \in \mathcal{M}_1(\Omega_T) | E_P(\xi) \le \hat{E}(\xi),\ \forall\ \xi \in \mathcal{H}^0_T\}$ is obviously the maximal one, which is convex and weak compact. However, by Choquet capacitability Theorem, all capacities induced by weak compact sets of probabilities in $\mathcal{A}$ are the same, i.e. $c_{\mathcal{P}} := \sup_{P\in\mathcal{P}} P = \sup_{P\in\mathcal{P}'} P =: c_{\mathcal{P}'}$ for any weak compact set $\mathcal{P}, \mathcal{P}' \in \mathcal{A}$. So we call it the capacity induced by $\hat{E}$. In fact, By Choquet capacitability Theorem, it suffices to prove the compact sets case. For any compact set $K \subset \Omega_T$, there exists an decreasing sequence $\{\varphi_n\} \subset C^+_b(\Omega_T)$ such that $1_K \le \varphi_n \le 1$ and $\varphi_n \downarrow 1_K$. Then by Theorem 28 in [DHP08],

$$c_{\mathcal{P}}(K) = \lim_{n\to\infty} \sup_{P\in\mathcal{P}} E_P(\varphi_n) = \lim_{n\to\infty} \hat{E}(\varphi_n) = \lim_{n\to\infty} \sup_{P\in\mathcal{P}'} E_P(\varphi_n) = c_{\mathcal{P}'}(K).$$

iii) All capacities induced by sets of probabilities in $\mathcal{A}$ are the same on open sets. In fact, for any $\mathcal{P} \in \mathcal{A}$, let $\bar{\mathcal{P}}$ be the weak closure of $\mathcal{P}$. Since $c_{\bar{\mathcal{P}}} = c_{\mathcal{P}}$ on open sets, we get the desired result by ii).

iv)Let $(\Omega^0, \{\mathcal{F}^0_t\}, \mathcal{F}, P^0)$ be a filtered probability space, and $\{W_t\}$ be a d-dimensional Brownian motion under P^0. [DHP08] proved that

$$\mathcal{P}'_M := \{P_0 \circ X^{-1} | X_t = \int_0^t h_s dW_s, h \in L^2_{\mathcal{F}}([0,T]; \Gamma^{1/2})\} \in \mathcal{A},$$

where $\Gamma^{1/2} := \{\gamma^{1/2} | \gamma \in \Gamma\}$ and Γ is the set in the representation of $G(\cdot)$.

v) Let $\mathcal{P}_M$ be the weak closure of $\mathcal{P}'_M$. Then under each $P \in \mathcal{P}_M$, the canonical process $B_t(\omega) = \omega_t$ for $\omega \in \Omega_T$ is a martingale. In fact, for any

$P \in \mathcal{P}_M$, there exists $\{P_n\} \subset \mathcal{P}'_M$ such that $P_n \to P$ weakly. For any $0 \le s \le t \le T$ and $\varphi \in C_b(\Omega_s)$, $E_{P_n}[\varphi(B)B_s] = E_{P_n}[\varphi(B)B_t]$ since $\{B_t\}$ is a martingale under P_n. Then by the integrability of B_t, B_s and the weak convergence of $\{P_n\}$ we have

$$E_P[\varphi(B)B_s] = \lim_n E_{P_n}[\varphi(B)B_s] = \lim_n E_{P_n}[\varphi(B)B_t] = E_P[\varphi(B)B_t].$$

Thus we get the desired result. $\square$

Definition 2.9 i) Let c be the capacity induced by $\hat{E}$. A map X on Ω_T with values in a topological space is said to be quasi-continuous w.r.t c if

$$\forall \varepsilon > 0, \textit{ there exists an open set } O \textit{ with } c(O) < \varepsilon \textit{ such that } X|_{O^c} \textit{ is continuous.}$$

ii) We say that $X : \Omega_T \to R$ has a quasi-continuous version if there exists a quasi-continuous function $Y : \Omega_T \to R$ with $X = Y$, c-q.s.. $\square$

By the definition of quasi-continuity and iii) in Remark 2.8, we know that the collections of quasi-continuous functions w.r.t. capacities induced by any set(not necessary weak compact) that represents $\hat{E}$ are the same.

Let $\|\varphi\|_{p,G} = [\hat{E}(|\varphi|^p)]^{1/p}$ for $\varphi \in C_b(\Omega_T)$, the completions of $C_b(\Omega_T)$, $\mathcal{H}_T^0$ and $L_{ip}(\Omega_T)$ under $\|\cdot\|_{p,G}$ are the same and denoted by $L_G^p(\Omega_T)$, where

$$L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, ..., B_{t_n}) | \forall n \ge 1, t_1, ..., t_n \in [0, T], \forall \varphi \in C_{b,Lip}(R^{d\times n})\}$$

and $C_{b,Lip}(R^{d\times n})$ denotes the set of bounded Lipschitz functions on $R^{d\times n}$.

Theorem 2.10[DHP08] For $p \ge 1$ the completion $L_G^p(\Omega_T)$ of $C_b(\Omega_T)$ is

$$L_G^p(\Omega_T) = \{X \in L^0 : X \textit{ has a q.c. version}, \lim_{n\to\infty} \hat{E}[|X|^p 1_{\{|X|>n\}}] = 0\},$$

where L^0 denotes the space of all R-valued measurable functions on Ω_T.

2.2 Basic notions on stochastic calculus in sublinear expectation space

Now we shall introduce some basic notions on stochastic calculus in sublinear expectation space $(\Omega_T, L_G^1, \hat{E})$. The canonical process $B_t(\omega) = \omega_t$ for $\omega \in \Omega_T$ is called G-Brownian motion.

For convenience of description, we only give the definition of Itô integral with respect to 1-dimensional G-Brownian motion. However, all results in sections 3-5 of this article hold for the d-dimensional case.

For $p \geq 1$, let $M_G^{p,0}(0,T)$ be the collection of processes in the following form: for a given partition $\{t_0, \cdots, t_N\} = \pi_T$ of $[0,T]$,

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) 1_{[t_j, t_{j+1})}(t),$$

where $\xi_i \in L_G^p(\Omega_{t_i})$, $i = 0, 1, 2, \cdots, N-1$. For each $\eta \in M_G^{p,0}(0,T)$, let $\|\eta\|_{M_G^p} = \{\hat{E} \int_0^T |\eta_s|^p ds\}^{1/p}$ and denote $M_G^p(0,T)$ the completion of $M_G^{p,0}(0,T)$ under norm $\|\cdot\|_{M_G^p}$.

Definition 2.11 For each $\eta \in M_G^{2,0}(0,T)$ with the form

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) 1_{[t_j, t_{j+1})}(t),$$

we define

$$I(\eta) = \int_0^T \eta(s) dB_s := \sum_{j=0}^{N-1} \xi_j (B_{t_{j+1}} - B_{t_j}).$$

The mapping $I : M_G^{2,0}(0,T) \to L_G^2(\Omega_T)$ is continuous and thus can be continuously extended to $M_G^2(0,T)$.

We denote for some $0 \leq s \leq t \leq T$, $\int_s^t \eta_u dB_u := \int_0^T 1_{[s,t]}(u) \eta_u dB_u$. We have the following properties:

Let $\eta, \theta \in M_G^2(0,T)$ and let $0 \leq s \leq r \leq t \leq T$. Then in $L_G^1(\Omega_T)$ we have

(i) $\int_s^t \eta_u dB_u = \int_s^r \eta_u dB_u + \int_r^t \eta_u dB_u$,

(ii)$\int_s^t (\alpha \eta_u + \theta_u) dB_u = \alpha \int_s^t \eta_u dB_u + \int_s^t \theta_u dB_u$, if α is bounded and in $L_G^1(\Omega_s)$,

(iii)$\hat{E}_t(X + \int_t^T \eta_s dB_s) = \hat{E}(X)$, $\forall X \in L_G^1(\Omega_T^t)$ and $\eta \in M_G^2(0,T)$.

Definition 2.12 Quadratic variation process of G-Brownian motion defined by

$$\langle B \rangle_t = B_t^2 - 2 \int_0^t B_s dB_s$$

is a continuous, nondecreasing process.

For $\eta \in M_G^{1,0}(0,T)$, define $Q_{0,T}(\eta) = \int_0^T \eta(s) d\langle B \rangle_s := \sum_{j=0}^{N-1} \xi_j (\langle B \rangle_{t_{j+1}} - \langle B \rangle_{t_j}) : M_G^{1,0}(0,T) \to L_G^1(\Omega_T)$. The mapping is continuous and can be extended to $M_G^1(0,T)$ uniquely.

Definition 2.13 A process $\{M_t\}$ with values in $L_G^1(\Omega_T)$ is called a G-martingale if $\hat{E}_s(M_t) = M_s$ for any $s \leq t$. If $\{M_t\}$ and $\{-M_t\}$ are both G-martingale, we call $\{M_t\}$ a symmetric G-martingale.

Definition 2.14 For two process $\{X_t\}, \{Y_t\}$ with values in $L_G^1(\Omega_T)$, we say $\{X_t\}$ is a version of $\{Y_t\}$ if

$$X_t = Y_t, \quad q.s. \;\; \forall t \in [0, T].$$

By the same arguments as in the classical linear case, for which we refer to [HWY92] for instance, we have the following lemma.

Lemma 2.15 Any symmetric G-martingale $\{M_t\}_{t\in[0,T]}$ has a RCLL(right continuous with left limit) version. $\square$

In the rest of this article, we only consider the RCLL versions of symmetric G-martingales.

Theorem 2.16 [P07] For each $x \in R$, $Z \in M_G^2(0,T)$ and $\eta \in M_G^1(0,T)$, the process

$$M_t = x + \int_0^t Z_s dB_s + \int_0^t \eta_s d\langle B\rangle_s - \int_0^t 2G(\eta_s)ds, \; t \in [0, T]$$

is a martingale. $\square$

Remark 2.17 Specially, $-K_t = \int_0^t \eta_s d\langle B\rangle_s - \int_0^t 2G(\eta_s)ds$ is a G-martingale, which is a surprising result because $-K_t$ is a continuous, non-increasing process. [P07] conjectured that any G-martingale has the above form and gave the following result. $\square$

Theorem 2.18 [P07] For all $\xi = \varphi(B_{t_1} - B_{t0}, \cdots, B_{t_n} - B_{t_{n-1}}) \in L_{ip}(\Omega_T)$, we have the following representation:

$$\xi = \hat{E}(\xi) + \int_0^T Z_t dB_t + \int_0^T \eta_t d\langle B\rangle_t - \int_0^T 2G(\eta_t)dt. \tag{2.2.1}$$

where $Z \in M_G^2(0,T)$ and $\eta \in M_G^1(0,T)$.

[STZ09] defined $\|\xi\|_{\mathcal{L}_2^0} = \{\hat{E}[\sup_{t\in[0,T]} \hat{E}_t(|\xi|^2)]\}^{1/2}$ on $L_{ip}(\Omega_T)$ and generalized the above result to the completion $\mathcal{L}_2^0$ of $L_{ip}(\Omega_T)$ under $\|\cdot\|_{\mathcal{L}_2^0}$.

Theorem 2.19 [STZ09] For all $\varphi \in \mathcal{L}_2^0$, there exists $\{Z_t\}_{t\in[0,T]} \in M_G^2(0,T)$ and a continuous increasing process $\{K_t\}_{t\in[0,T]}$ with $K_0 = 0, K_T \in L_G^2(\Omega_T)$ and $\{-K_t\}_{t\in[0,T]}$ a G-martingale such that

$$X_t := \hat{E}_t(\varphi) = \hat{E}(\varphi) + \int_0^t Z_s dB_s - K_t =: M_t - K_t, \; q.s. \tag{2.2.2}$$

$\square$

3 G-evaluation

In this section, we introduce an sublinear expectation which is induced by G-expectation and investigate some of its properties.

For $\xi \in \mathcal{H}_T^0$, let $\mathcal{E}(\xi) = \hat{E}[\sup_{u\in[0,T]} \hat{E}_u(\xi)]$ for all $\xi \in \mathcal{H}_T^0$ where $\hat{E}$ is the G-expectation. For convenience, we call $\mathcal{E}$ G-evaluation. First we give the following representation for G-evaluation, which is similar to that of G-expectation.

Theorem 3.1 There exists a weak compact subset $\mathcal{P}^{\mathcal{E}} \subset \mathcal{M}_1(\Omega)$ such that

$$\mathcal{E}(\xi) = \max_{P\in\mathcal{P}^{\mathcal{E}}} E_P(\xi) \ \text{ for } \ \text{all } \xi \in \mathcal{H}_T^0.$$

Proof. 1. Obviously, $(\Omega, \mathcal{H}_T^0, \mathcal{E})$ is a sublinear expectation space. Then there exists a family of positive linear functionals $\mathcal{I}$ on $\mathcal{H}_T^0$ such that

$$\mathcal{E}(\xi) = \max_{I\in\mathcal{I}} I(\xi) \ \text{ for } \ \text{all } \xi \in \mathcal{H}_T^0.$$

2. In the following, we give some calculations.

For any $0 \le s \le t \le T$ and $u \in [0,T]$,

$$\hat{E}_u|B_t - B_s|^{2n} \le \begin{cases} |B_t - B_s|^{2n}, & \text{if } \ u \ge t, \\ c_n(t-s)^n, & \text{if } \ u \le s, \\ 2^{2^n-1}[c_n(t-u)^n + |B_u - B_s|^{2n}], & \text{if } \ s < u < t. \end{cases}$$

Thus $\hat{E}_u|B_t - B_s|^{2n} \le 2^{2^n-1}[c_n(t-s)^n + \sup_{u\in[s,t]} |B_u - B_s|^{2n}]$, and by B-D-G inequality

$\mathcal{E}|B_t - B_s|^{2n} \le 2^{2^n-1}[c_n(t-s)^n + b_n(t-s)^n] =: d_n(t-s)^n.$

3. Noting the discussion in Remark 2.8, we can prove the desired representation by just the same arguments as in [HP09]. $\square$

For $p \ge 1$ and $\xi \in \mathcal{H}_T^0$, define $\|\xi\|_{p,\mathcal{E}} = [\mathcal{E}(|\xi|^p)]^{1/p}$ and denote $L_{\mathcal{E}}^p(\Omega_T)$ the completion of $\mathcal{H}_T^0$ under $\|\cdot\|_{p,\mathcal{E}}$.

We shall give an estimate between the two norms $\|\cdot\|_{p,\mathcal{E}}$ and $\|\cdot\|_{p,G}$. As the substitute of Doob's maximal inequality, the estimate will play a critical role in the proof to the martingale decomposition theorem in the next section. First, we shall give a lemma.

For convenience, we say ξ is symmetric if $\xi \in L_G^1$ with $\hat{E}(\xi) + \hat{E}(-\xi) = 0$.

Lemma 3.2 For $\xi \in L_{ip}(\Omega_T)$, there exists nonnegative $K_T \in L^1_G(\Omega_T)$ such that $\xi + K_T$ is symmetric. Moreover, for any $1 < \gamma < \beta$, $\gamma \leq 2$, $K_T \in L^\gamma_G(\Omega_T)$ and

$$\|K_T\|^\gamma_{L^\gamma_G} \leq 14 C^\gamma_{\beta/\gamma} \|\xi\|^\beta_{L^\beta_G},$$

where $C_{\beta/\gamma} = \sum_{i=1}^\infty i^{-\beta/\gamma}$.

Proof. Let $\xi^n = (\xi \wedge n) \vee (-n)$ and $\eta^n = \xi^{n+1} - \xi^n$ for $n \geq 0$. Then by Theorem 2.18, for each n, we have the following representation (2.2.1):

$$X^n_t := \hat{E}_t(\eta^n) = M^n_t - K^n_t,$$

where $\{M^n_t\}$ is a symmetric G-martingale with $M^n_T \in L^2_G(\Omega_T)$ and $\{K^n_t\}$ is a continuous increasing process with $K^n_0 = 0, K^n_T \in L^2_G(\Omega_T)$ and $\{-K^n_t\}$ a G-martingale. Fix $P \in \mathcal{P}_M$. By $It\hat{o}'s$ formula

$$(\eta^n)^2 = 2\int_0^T X^n_t dX^n_t + [M^n]_T, \; P - a.s.$$

Take expectation under P, we have

$$E_P[(M^n_T)^2] \leq E_P[(\eta^n)^2] + 2E_P(K^n_T).$$

Take supremum over $\mathcal{P}_M$, we have

$$\hat{E}[(M^n_T)^2] \leq \hat{E}[(\eta^n)^2] + 2\hat{E}(K^n_T) \leq 5\hat{E}(|\eta^n|).$$

Therefore,

$$\hat{E}[(K^n_T)^2] \leq 2(\hat{E}[(\eta^n)^2] + \hat{E}[(M^n_T)^2]) \leq 12\hat{E}(|\eta^n|).$$

Consequently, for any $1 < \gamma < \beta$ and $\gamma \leq 2$

$$\hat{E}[(K^n_T)^\gamma] \leq \hat{E}(K^n_T) + \hat{E}[(K^n_T)^2]) \leq 14\hat{E}(|\eta^n|).$$

$$\begin{aligned}
& \hat{E}[(\sum_{i=n+1}^{n+m} K^i_T)^\gamma] \\
\leq \; & [\sum_{i=n+1}^{n+m} i^{-\beta/\gamma}]^{\gamma-1} \sum_{i=n+1}^{n+m} i^{\beta/\gamma^*} \hat{E}[(K^i_T)^\gamma] \\
\leq \; & 14 C^{\gamma-1}_{\beta/\gamma}(n,m) \sum_{i=n+1}^{n+m} i^{\beta/\gamma^*} \hat{E}(|\eta^i|) \\
\leq \; & 14 C^{\gamma-1}_{\beta/\gamma}(n,m) \sum_{i=n+1}^{n+m} i^{\beta/\gamma^*} c(|\xi| > i) \\
\leq \; & 14\hat{E}(|\xi|^\beta) C^\gamma_{\beta/\gamma}(n,m).
\end{aligned}$$

where $C_{\beta/\gamma}(n,m) = \sum_{i=n+1}^{n+m} i^{-\beta/\gamma}, \gamma^* = \gamma/(\gamma-1)$.

So $\{\sum_{n=0}^{N} K_T^n\}$ is a Cauchy sequence in $L_G^\gamma(\Omega_T)$. Let $K_T := \lim_{L_G^\gamma, N\to\infty} \sum_{n=0}^{N} K_T^n$, then $\|K_T\|_{L_G^\gamma}^\gamma \leq 14 C_{\beta/\gamma}^\gamma \|\xi\|_{L_G^\beta}^\beta$. Since $\eta^n + K_T^n$ is symmetric for each $n \geq 0$, $\xi^N + \sum_{n=0}^{N-1} K_T^n$ is symmetric for each $N \geq 1$. Consequently, $\xi + K_T$ is symmetric. □

Theorem 3.3 For any $\alpha \geq 1$ and $\delta > 0$, $L_G^{\alpha+\delta}(\Omega_T) \subset L_{\mathcal{E}}^\alpha(\Omega_T)$. More precisely, for any $1 < \gamma < \beta := (\alpha+\delta)/\alpha$, $\gamma \leq 2$, we have

$$\|\xi\|_{\alpha,\mathcal{E}}^\alpha \leq \gamma^*\{\|\xi\|_{\alpha+\delta,G}^\alpha + 14^{1/\gamma} C_{\beta/\gamma} \|\xi\|_{\alpha+\delta,G}^{(\alpha+\delta)/\gamma}\},\ \forall \xi \in L_{ip}(\Omega_T),$$

where $C_{\beta/\gamma} = \sum_{i=1}^{\infty} i^{-\beta/\gamma}, \gamma^* = \gamma/(\gamma-1)$.

Proof. For $\xi \in L_{ip}(\Omega_T)$, $|\xi|^\alpha \in L_{ip}(\Omega_T)$. By Lemma 3.2, there exists $K_T \in L_G^1(\Omega_T)$ such that for any $1 < \gamma < \beta$, $\gamma \leq 2$ $K_T \in L_G^\gamma(\Omega_T)$ and $M_T := |\xi|^\alpha + K_T$ is symmetric. Let $M_t = \hat{E}_t(M_T)$. Then

$$\begin{aligned}
\|\xi\|_{\alpha,\mathcal{E}}^\alpha &= \hat{E}[\sup_{t\in[0,T]} \hat{E}_t(|\xi|^\alpha)] \\
&\leq \hat{E}(\sup_{t\in[0,T]} M_t) \\
&\leq [\hat{E}(\sup_{t\in[0,T]} M_t^\gamma)]^{1/\gamma} \\
&\leq \gamma^* \|M_T\|_{\gamma,G} \\
&\leq \gamma^*\{\||\xi|^\alpha\|_{\gamma,G} + \|K_T\|_{\gamma,G}\} \\
&\leq \gamma^*\{\|\xi\|_{\alpha+\delta,G}^\alpha + 14^{1/\gamma} C_{\beta/\gamma} \|\xi\|_{\alpha+\delta,G}^{(\alpha+\delta)/\gamma}\}.
\end{aligned}$$

□

Let $\mathcal{P}_\mathcal{E}$ be weak compact subsets of $\mathcal{M}_1(\Omega_T)$ which represent $\mathcal{E}$. Define capacity $c_\mathcal{E}(A) = \sup_{P\in\mathcal{P}_\mathcal{E}}(A)$. We all $c_\mathcal{E}$ the capacity induced by $\mathcal{E}$.

By the above estimate, we can get the following equivalence between the Choquet capacities induced by $\hat{E}$ and $\mathcal{E}$.

Corollary 3.4 There exists $C > 0$ such that for any set $A \in \mathcal{B}(\Omega_T)$, $c(A)^2 \leq c_\mathcal{E}(A)^2 \leq Cc(A)$.

Proof. By Choquet capacitability Theorem, it suffices to prove the compact sets case. For any compact set $K \subset \Omega_T$, there exists an decreasing sequence $\{\varphi_n\} \subset C_b^+(\Omega_T)$ such that $1_K \leq \varphi_n \leq 1$ and $\varphi_n \downarrow 1_K$. Let $\alpha = \delta = 1$ in the above Theorem 3.3, there exists $1 < \gamma < 2$ and $C > 0$, such that

$$[\mathcal{E}(\varphi_n)]^2 \leq C\hat{E}(\varphi_n).$$

Then by Theorem 28 in [DHP08],

$$c_{\mathcal{E}}(K)^2 \leq Cc(K).$$

$\square$

Corollary 3.5 The collections of quasi-continuous functions on Ω_T w.r.t c and $c_{\mathcal{E}}$ are the same. $\square$

4 Applications to G-martingale decomposition

4.1 Generalized It$\hat{o}$ integral

Let $H_G^0(0,T)$ be the collection of processes in the following form: for a given partition $\{t_0, \cdots, t_N\} = \pi_T$ of $[0,T]$,

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) 1_{[t_j,t_{j+1})}(t),$$

where $\xi_i \in L_{ip}(\Omega_{t_i})$, $i = 0, 1, 2, \cdots, N-1$. For each $\eta \in H_G^0(0,T)$ and $p \geq 1$, let $\|\eta\|_{H_G^p} = \{\hat{E}(\int_0^T |\eta_s|^2 ds)^{p/2}\}^{1/p}$ and denote $H_G^p(0,T)$ the completion of $H_G^0(0,T)$ under norm $\|\cdot\|_{H_G^p}$. It's easy to prove that $H_G^2(0,T) = M_G^2(0,T)$.

Definition 4.1 For each $\eta \in H_G^0(0,T)$ with the form

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) 1_{[t_j,t_{j+1})}(t),$$

we define

$$I(\eta) = \int_0^T \eta(s) dB_s := \sum_{j=0}^{N-1} \xi_j (B_{t_{j+1}} - B_{t_j}).$$

By B-D-G inequality, the mapping $I : H_G^0(0,T) \to L_G^p(\Omega_T)$ is continuous under $\|\cdot\|_{H_G^p}$ and thus can be continuously extended to $H_G^p(0,T)$.

4.2 G-martingale decomposition

Let $\mathcal{B}_t = \sigma\{B_s | s \leq t\}$, $\mathcal{F}_t = \cap_{r>t} \mathcal{B}_r$ and $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$. $\tau : \Omega_T \to [0,T]$ is called a $\mathbb{F}$ stopping time if $[\tau \leq t] \in \mathcal{F}_t$, $\forall t \in [0,T]$.

In order to prove the more general G-martingale decomposition decomposition theorem, we first introduce a famous lemma, for which we refer to [RY94].

Definition 4.3. A positive, adapted right-continuous process X is dominated by an increasing process A with $A_0 \geq 0$ if

$$E[X_\tau] \leq E[A_\tau]$$

for any bounded stopping time τ.

Lemma 4.4. If X is dominated by A and A is continuous, for any $k \in (0,1)$

$$E[(X_T^*)^k] \leq \frac{2-k}{1-k} E[A_T^k],$$

where $X_T^* = \sup_{t\in[0,T]} X_t$.

Theorem 4.5. For $\xi \in L_G^\beta(\Omega_T)$ with some $\beta > 1$, $X_t = \hat{E}_t(\xi)$, $t \in [0,T]$ has the following decomposition:

$$X_t = X_0 + \int_0^t Z_s dB_s - K_t, \; q.s.$$

where $\{Z_t\} \in H_G^1(0,T)$ and $\{K_t\}$ is a continuous increasing process with $K_0 = 0$ and $\{-K_t\}_{t\in[0,T]}$ a G-martingale. Furthermore, the above decomposition is unique and $\{Z_t\} \in H_G^\alpha(0,T)$, $K_T \in L_G^\alpha(\Omega_T)$ for any $1 \leq \alpha < \beta$.

Proof. For $\xi \in L_G^\beta(\Omega_T)$, there exists a sequence $\{\xi^n\} \subset L_{ip}(\Omega_T)$ such that $\|\xi^n - \xi\|_{\beta,G} \to 0$. By Theorem 2.18, we have the following decomposition

$$X_t^n := \hat{E}_t(\xi^n) = X_0 + \int_0^t Z_s^n dB_s - K_t := M_t^n - K_t^n, \; q.s.$$

where $\{Z_t^n\} \in H_G^2(0,T)$ and $\{K_t^n\}$ is a continuous increasing process with $K_0^n = 0$ and $\{-K_t^n\}_{t\in[0,T]}$ a G-martingale.

Fix $P \in \mathcal{P}_M$, by Itô$'s$ formula,

$$(X_\tau^n)^2 = 2\int_0^\tau X_s^n dX_s^n + [M^n]_\tau, \; \forall \text{ stopping time } \tau.$$

Take expectation under P, we have

$$\begin{aligned} E_P[(M_\tau^n)^2] &= E_P[(X_\tau^n)^2] + 2E_P(\int_0^\tau X_s^n dK_s^n) \\ &\leq E_P[(X_\tau^{n*})^2] + 2E_P(\int_0^\tau X_s^{n*} dK_s^n), \end{aligned}$$

where $X_t^{n*} = \sup_{0<s\leq t} |X_s^n|$.

In the following, C_α will always designate a universal constant, which may vary from line to line.

$\beta \leq 2$ case.

Consequently, for any $1 < \alpha < \beta$, by Lemma 4.4

$$\begin{aligned} E_P[(M_T^{n*})^\alpha] &\leq C_\alpha\{E_P[(X_T^{n*})^\alpha] + E_P[(X_T^{n*})^{\alpha/2}(K_T^n)^{\alpha/2}]\} \\ &\leq C_\alpha\{E_P[(X_T^{n*})^\alpha] + \{E_P[(X_T^{n*})^\alpha]\}^{1/2}\{E_P[(K_T^n)^\alpha]\}^{1/2}\}, \end{aligned}$$

where $M_T^{n*} = \sup_{0<s\leq T} |M_s^n|$.

On the other hand,

$$(K_\tau^n)^2 \leq 2[(X_\tau^n)^2 + (M_\tau^n)^2], \ \forall \tau.$$

So

$$\begin{aligned} E_P[(K_\tau^n)^2] &\leq 2E_P[(X_\tau^n)^2 + (M_\tau^n)^2] \\ &\leq 2E_P[(X_\tau^{n*})^2 + (M_\tau^{n*})^2]. \end{aligned}$$

By this, we have

$$E_P[(K_T^n)^\alpha] \leq C_\alpha E_P[(X_T^{n*})^\alpha + (M_T^{n*})^\alpha].$$

So

$$\begin{aligned} & E_P[(M_T^{n*})^\alpha] \\ \leq\ & C_\alpha E_P[(X_T^{n*})^\alpha] + C_\alpha\{E_P[(X_T^{n*})^\alpha]\}^{1/2}\{E_P[(M_T^{*n})^\alpha]\}^{1/2} \\ \leq\ & 1/2C_\alpha E_P[(X_T^{n*})^\alpha] + 1/2E_P[(M_T^{n*})^\alpha]. \end{aligned}$$

Now, we have $E_P(|M_T^{n*}|^\alpha) \leq C_\alpha E_P[(X_T^{n*})^\alpha]$ and $E_P(|K_T^n|^\alpha) \leq C_\alpha E_P[(X_T^{n*})^\alpha]$.

Let $\widehat{X}_t := X_t^n - X_t^m$, $\widehat{M_t} := M_t^n - M_t^m$, $\widehat{K}_t := K_t^n - K_t^m$ and $\widetilde{K}_t := K_t^n + K_t^m$. By It$\hat{o}'s$ formula,

$$\widehat{X}_\tau^2 = 2\int_0^\tau \widehat{X}_s d\widehat{X}_s + [\widehat{M}]_\tau, \ \forall \tau.$$

Take expectation under P, we have

$$\begin{aligned} E_P[(\widehat{M}_\tau)^2] &= E_P[(\widehat{X}_\tau)^2] + 2E_P(\int_0^\tau \widehat{X}_s d\widehat{K}_s) \\ &\leq E_P[(\widehat{X}_\tau^*)^2] + 2E_P(\int_0^\tau \widehat{X}_s^* d\widetilde{K}_s), \end{aligned}$$

where $\widehat{X}_t^* = \sup_{0<s\le t} |\widehat{X}_s|$.

By the same arguments as above,

$$\begin{aligned} E_P[(\widehat{M}_T^*)^\alpha] &\le C_\alpha\{E_P[(\widehat{X}_T^*)^\alpha] + E_P[(\widehat{X}_T^*)^{\alpha/2}(\widetilde{K}_T)^{\alpha/2}]\} \\ &\le C_\alpha\{E_P[(\widehat{X}_T^*)^\alpha] + \{E_P[(\widehat{X}_T^*)^\alpha]\}^{1/2}\{E_P[(\widetilde{K}_T)^\alpha]\}^{1/2}\}, \end{aligned}$$

where $\widehat{M}_T^* = \sup_{0<s\le T} |\widehat{M}_s|$.

Take supremum over $\mathcal{P}_M$, we get

$$\hat{E}(|K_T^n|^\alpha) \le C_\alpha \hat{E}[(X_T^{n*})^\alpha]$$

and

$$\hat{E}[(\widehat{M}_T^*)^\alpha] \le C_\alpha\{\hat{E}[(\widehat{X}_T^*)^\alpha] + \{\hat{E}[(\widehat{X}_T^*)^\alpha]\}^{1/2}\{\hat{E}[(\widetilde{K}_T)^\alpha]\}^{1/2}\}.$$

By Theorem 3.3, $\sup_n \hat{E}[(X_T^{n*})^\alpha] < \infty$ and $\sup_{n,m\ge N} \hat{E}[(\widehat{X}_T^*)^\alpha] \to 0$ as N goes to infinity. Therefore, $\sup_{n,m\ge N} \hat{E}[(\widehat{M}_T^*)^\alpha] \to 0$ and consequently

$$\sup_{n,m\ge N} \hat{E}(\sup_{t\in[0,T]} |K_t^n - K_t^m|^\alpha) \to 0$$

as N goes to infinity. Then there exists symmetric G-martingale $\{M_t\}$ and a process $\{K_t\}$ valued in $L_G^\alpha(\Omega_T)$ such that

$$\hat{E}(\sup_{t\in[0,T]} |M_t^n - M_t|^\alpha) \to 0$$

and

$$\hat{E}(\sup_{t\in[0,T]} |K_t^n - K_t|^\alpha) \to 0$$

as n goes to infinity.

So by B-D-G inequality, there exists $\{Z_t\} \in H_G^\alpha(0,T)$ such that $\|Z - Z^n\|_{H_G^\alpha} \to 0$.

Consequently,

$$X_t = \lim_{L_G^\alpha, n\to\infty} X_t^n = \lim_{L_G^\alpha, n\to\infty} \int_0^t Z_s^n dB_s - \lim_{L_G^\alpha, n\to\infty} K_t^n = \int_0^t Z_s dB_s - K_t.$$

$\beta > 2$ case.

For $2 < \alpha < \beta$,

$$[M^n]_T^{\alpha/2} \le C_\alpha(|X_T^n|^\alpha + |\int_0^T X_s^n dK_s^n|^{\alpha/2} + |\int_0^T X_s^n dM_s^n|^{\alpha/2}).$$

So

$$
\begin{aligned}
& E_P([M^n]_T^{\alpha/2}) \\
\leq\ & C_\alpha[E_P(|X_T^n|^\alpha) + E_P(|\int_0^T X_s^n dK_s^n|^{\alpha/2}) + E_P(|\int_0^T X_s^n dM_s^n|^{\alpha/2})] \\
\leq\ & C_\alpha\{E_P(|X_T^n|^\alpha) + \{E_P[(X_T^{n*})^\alpha]\}^{1/2}\{E_P[(K_T^n)^\alpha]\}^{1/2} \\
& +\{E_P[(X_T^{n*})^\alpha]\}^{1/2}\{E_P([M^n]_T^{\alpha/2})\}^{1/2}\}.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
E_P[(K_T^n)^\alpha] &\leq C_\alpha[E_P(|X_T^n|^\alpha) + E_P(|M_T^n|^\alpha)] \\
&\leq C_\alpha\{E_P[(X_T^{n*})^\alpha] + E_P([M^n]^{\alpha/2})\}.
\end{aligned}
$$

Therefore,

$$E_P([M^n]_T^{\alpha/2}) \leq C_\alpha E_P[(X_T^{n*})^\alpha]$$

and

$$E_P[(K_T^n)^\alpha] \leq C_\alpha E_P[(X_T^{n*})^\alpha].$$

By the same arguments, we get

$$E_P([\widehat{M}]_T^{\alpha/2}) \leq C_\alpha\{E_P[(\widehat{X}_T^*)^\alpha] + \{E_P[(\widehat{X}_T^*)^\alpha]\}^{1/2}\{E_P[(\widetilde{K}_T)^\alpha]\}^{1/2}\}.$$

The rest of the proof is just similar to the $\beta \leq 2$ case.

□

Theorem 4.6 Let $\xi \in L_G^\beta(\Omega_T)$ for some $\beta > 1$ with $\hat{E}(\xi) + \hat{E}(-\xi) = 0$, then there exists $\{Z_t\}_{t\in[0,T]} \in H_G^1(0,T)$ such that

$$\xi = \hat{E}(\xi) + \int_0^T Z_s dB_s.$$

Furthermore, the above representation is unique and $\{Z_t\} \in H_G^\alpha(0,T)$ for any $1 \leq \alpha < \beta$.

Proof. By Theorem 4.5, for $\xi \in L_G^\beta(\Omega_T)$ with some $\beta > 1$, $X_t = \hat{E}_t(\xi)$, $t \in [0,T]$ has the following decomposition:

$$\xi = \hat{E}(\xi) + \int_0^T Z_s dB_s - K_T,\ q.s.$$

where $\{Z_t\} \in H_G^\alpha(0,T)$ for any $1 \leq \alpha < \beta$ and $\{K_t\}$ is a continuous increasing process with $K_0 = 0$ and $\{-K_t\}_{t\in[0,T]}$ a G-martingale. If ξ is symmetric in addition, then

$$\hat{E}(K_T) = \hat{E}(\xi) + \hat{E}(-\xi) = 0.$$

So $K_T = 0, q.s.$ and

$$\xi = \hat{E}(\xi) + \int_0^T Z_s dB_s.$$

□

Remark 4.7 Since $\xi \in L_G^\beta(\Omega_T)$, we have by B-D-G and Doob's maximal inequality

$$\{\hat{E}(\int_0^T |Z_s|^2 ds)^{\beta/2}\}^{1/\beta} < \infty.$$

But we still can't say that $\{Z_t\}_{t\in[0,T]} \in H_G^\beta(0,T)$. Fortunately, by a stopping time technique, Corollary 5.2 in [S10] shows that $\{Z_t\}_{t\in[0,T]}$ does belong to $H_G^\beta(0,T)$ for $\xi \in L_G^\beta(\Omega_T)$. Here we will give a direct proof for $\beta = 2$ case.

Theorem 4.8 Let $\xi \in L_G^2(\Omega_T)$ with $\hat{E}(\xi) + \hat{E}(-\xi) = 0$, then there exists $\{Z_t\}_{t\in[0,T]} \in M_G^2(0,T)$ such that

$$\xi = \hat{E}(\xi) + \int_0^T Z_s dB_s.$$

Proof. Let $\xi^n = (\xi \wedge n) \vee (-n)$, then $\hat{E}[(\xi - \xi^n)^2] \to 0$. Let $M_t^n = \hat{E}_t(\xi^n)$ and $-\widetilde{M}_t^n = \hat{E}_t(-\xi^n)$. By Theorem 4.5, there exist $\{Z_t^n\}, \{\widetilde{Z}_t^n\} \in M_G^2(0,T)$ and continuous increasing processes $\{K_t^n\}_{t\in[0,T]}, \{\widetilde{K}_t^n\}_{t\in[0,T]}$ with $K_0^n = \tilde{K}_0^n = 0$ and $\{-K_t^n\}_{t\in[0,T]}$, $\{-\widetilde{K}_t^n\}_{t\in[0,T]}$ G-martingales such that

$$\begin{aligned} M_t^n &= M_0^n + \int_0^t Z_s^n dB_s - K_t^n =: N_t^n - K_t^n, \\ \widetilde{M}_t^n &= \widetilde{M}_0^n - \int_0^t Z_s^n dB_s + \widetilde{K}_t^n =: \widetilde{N}_t^n + \widetilde{K}_t^n. \end{aligned}$$

Let

$$\hat{M}_t^n := M_t^n - \widetilde{M}_t^n = N_t^n - \widetilde{N}_t^n - (\widetilde{K}_t^n + K_t^n) =: \hat{N}_t^n - \hat{K}_t^n.$$

Fix $P \in \mathcal{P}_M$. By $\text{It}\hat{o}'s$ formula,

$$0 = (\hat{M}_T^n)^2 = 2\int_0^T \hat{M}_s^n d\hat{M}_s^n + [\hat{N}^n]_T, \ P - a.s. \ \forall t \in [0,T].$$

Take expectation in the above equation,

$$E_P[(\hat{N}_T^n)^2] = 2E_P(\int_0^T \hat{M}_s^n d\hat{K}_s^n) \le 4nE_P[\hat{K}_T^n].$$

So $\hat{E}[(\hat{N}_T^n)^2] \leq 4n\hat{E}[\hat{K}_T^n]$. Noting that

$$\begin{aligned}
& \hat{E}[\hat{K}_T^n] \\
= & \hat{E}[\hat{N}_T^n] + \hat{E}[-\hat{M}_T^n] \\
= & \hat{E}[\xi^n] + \hat{E}[-\xi^n] \\
= & \hat{E}[\xi^n - \xi] + \hat{E}[\xi] + \hat{E}[-(\xi^n - \xi)] - \hat{E}[\xi] \\
\leq & 2\hat{E}[|\xi^n - \xi|] \\
\leq & 2\hat{E}[|\xi| 1_{[|\xi|>n]}],
\end{aligned}$$

we get

$$\hat{E}[(\hat{N}_T^n)^2] \leq 4n\hat{E}[\hat{K}_T^n] \leq 8n\hat{E}[|\xi| 1_{[|\xi|>n]}] \leq 8\hat{E}[|\xi|^2 1_{[|\xi|>n]}] \to 0.$$

So

$$\hat{E}[(K_T^n)^2] \leq \hat{E}[(\hat{K}_T^n)^2] = \hat{E}[(\hat{N}_T^n)^2] \to 0.$$

Let $X^n := \xi^n - \xi = N_T^n - \xi - K_T^n$. Then

$$\hat{E}[(N_T^n - \xi)^2] \leq 2\{\hat{E}[(X^n)^2] + \hat{E}[(K_T^n)^2]\} = 2\{\hat{E}[(\xi^n - \xi)^2] + \hat{E}[(K_T^n)^2]\} \to 0.$$

Since $\{\eta \in L_G^2(\Omega_T) | \ \eta = \int_0^T Z_s dB_s$ for some $Z \in M_G^2(0,T)\}$ is closed in $L_G^2(\Omega_T)$, we proved the desired result. $\Box$

By the proof in Theorem 4.5, we can get the following estimates, which may be useful in the follow-up work of G-martingale theory.

Corollary 4.9 For $\xi, \xi' \in L_G^\beta(\Omega_T)$ with some $\beta > 1$, let $\xi = M_T - K_T$ and $\xi' = M_T' - K_T'$ be the decomposition in Theorem 4.5. Then for any $1 < \alpha < \beta$, $1 < \gamma < \beta/\alpha, \gamma \leq 2$, there exists $C_{\alpha,\beta,\gamma}$ such that

$$\|K_T\|_{\alpha,G}^\alpha \leq C_{\alpha,\beta,\gamma}\{\|\xi\|_{\beta,G}^\alpha + \|\xi\|_{\beta,G}^{\beta/\gamma}\}$$

and

$$\begin{aligned}
& \|M_T - M_T'\|_{\alpha,G}^\alpha \leq C_{\alpha,\beta,\gamma}\{\|\xi - \xi'\|_{\beta,G}^\alpha + \|\xi - \xi'\|_{\beta,G}^{\beta/\gamma}\} \\
+ & C_{\alpha,\beta,\gamma}\{\|\xi - \xi'\|_{\beta,G}^\alpha + \|\xi - \xi'\|_{\beta,G}^{\beta/\gamma}\}^{1/2}\{1 + \|\xi\|_{\beta,G}^{\beta/\gamma} + \|\xi'\|_{\beta,G}^{\beta/\gamma}\}^{1/2}.
\end{aligned}$$

5 Regular properties for G-martingale

Definition 5.1 We say that a process $\{M_t\}$ with values in $L_G^1(\Omega_T)$ is quasi-continuous if

$\forall \varepsilon > 0$, *there exists open set* G *with* $c(G) < \varepsilon$ *such that* $M_\cdot(\cdot)$ *is continuous on* $G^c \times [0, T]$.

Corollary 5.2 Any G-martingale $\{M_t\}$ with $M_T \in L_G^\beta(\Omega_T)$ for some $\beta > 1$ has a quasi-continuous version.

Proof. For $\xi \in L_{ip}(\Omega_T)$, $M_t = \hat{E}_t(\xi)$ is continuous on $[0,T] \times \Omega_T$. In fact, for $\xi = \varphi(B_{t_1} - B_{t_0}, \cdots, B_{t_n} - B_{t_n - 1})$ with $\varphi \in C_{b,lip}(R^n)$, $M_t(\cdot)$ is obvious continuous on Ω_T for fixed $t \in [0,T]$. On the other hand, fix $\omega \in \Omega_T$

$$\begin{aligned} & |M_{t_n}(\omega) - M_{t_{n-1}}(\omega)| \\ \leq\ & |\varphi(x_1, \cdots, x_{n-1}, x_n) - \hat{E}[\varphi(x_1, \cdots, x_{n-1}, B_{t_n} - B_{t_n - 1})]| \\ \leq\ & L\hat{E}(|B_{t_n} - B_{t_n - 1} - x_n|) \\ \leq\ & L(\bar{\sigma}(t_n - t_{n-1})^{1/2} + |\omega_{t_n} - \omega_{t_{n-1}}|), \end{aligned}$$

where L is the Lipschitz constant of φ and $x_i = B_{t_i}(\omega) - B_{t_{i_1}}(\omega)$. In fact, the above estimate holds for any $s, t \in [t_{i-1}, t_i]$ for some $1 \leq i \leq n$. Then for any $s_k \to s$ and $\omega^k \to \omega$,

$$\begin{aligned} & |M_{s_k}(\omega^k) - M_s(\omega)| \\ \leq\ & |M_{s_k}(\omega^k) - M_s(\omega^k)| + |M_s(\omega^k) - M_s(\omega)| \\ \leq\ & L(\bar{\sigma}|s_k - s|^{1/2} + |\omega^k_{s_k} - \omega^k_s|) + |M_s(\omega^k) - M_s(\omega)| \to 0. \end{aligned}$$

For $\xi = M_T \in L_G^\beta(\Omega_T)$, by Theorem 3.3, there exists $1 < \alpha < \beta$ and $\{\xi^n\} \subset L_{ip}(\Omega_T)$ such that $\|\xi^n - \xi\|_{\mathcal{E},\alpha} \to 0$. Let $M_t^n := \hat{E}_t(\xi^n)$. Then

$$\sup_{m>n} \hat{E}[\sup_{t\in[0,T]} (|M_t^m - M_t^n|] \leq \sup_{m>n} \|\xi^n - \xi^m\|_{\mathcal{E},\alpha} \to 0$$

as n goes to infinity. So there exists a subsequence $\{n_k\}$ such that

$$\sup_{m>n_k} \hat{E}[\sup_{t\in[0,T]} (|M_t^m - M_t^{n_k}|] < 1/4^k.$$

Consequently,

$$\hat{E}[\sum_{k=1}^{\infty} \sup_{t\in[0,T]} (|M_t^{n_{k+1}} - M_t^{n_k}|)] \leq \sum_{k=1}^{\infty} \hat{E}[\sup_{t\in[0,T]} (|M_t^{n_{k+1}} - M_t^{n_k}|)] < \infty.$$

Then there exists $\{\widetilde{M}_t\}$ such that

$$\sup_{t\in[0,T]} (|M_t^{n_k} - \widetilde{M}_t|) \leq \sum_{i=k}^{\infty} \sup_{t\in[0,T]} (|M_t^{n_{i+1}} - M_t^{n_i}|),\ \forall k \geq 1.$$

For any $\varepsilon > 0$, let $O_k^\varepsilon := [\sup_{t\in[0,T]}(|M_t^{n_{k+1}} - M_t^{n_k}|) > 1/(2^k\varepsilon)]$ and $O^\varepsilon = \cup_{k=1}^\infty O_k^\varepsilon$. Then $c(O^\varepsilon) \leq \sum_{k=1}^\infty c(O_k^\varepsilon) < \varepsilon$ and on $(O^\varepsilon)^c$

$$\sup_{t\in[0,T]}(|M_t^{n_k} - \widetilde{M}_t|) \leq \sum_{i=k}^\infty \sup_{t\in[0,T]}(|M_t^{n_{i+1}} - M_t^{n_i}|) \leq 1/(2^{k-1}\varepsilon), \ \forall k \geq 1.$$

So

$$\sup_{\omega\in(O^\varepsilon)^c}\sup_{t\in[0,T]}(|M_t^{n_k} - \widetilde{M}_t|) \to 0$$

and $\{\widetilde{M}_t\}$ is a quasi-continuous version of $\{M_t\}$. $\Box$

Theorem 5.3 Any G-martingale $\{M_t\}$ has a quasi-continuous version.

Proof. Let $\xi := M_T$ and $\xi^n = (\xi \wedge n) \vee (-n)$. For $m > n$, let $\{X_t^n\}, \{X_t^m\}, \{X_t^{n,m}\}$ be the quasi-continuous versions of $\{\hat{E}_t(\xi^n)\}, \{\hat{E}_t(\xi^m)\}$, $\{\hat{E}_t(|\xi^n - \xi^m|)\}$ respectively.

We claim that

$$\hat{E}[\sup_{m>n}\sup_{t\in[0,T]}(|X_t^m - X_t^n| \wedge 1)] \leq \hat{E}[\sup_{m>n}\sup_{t\in[0,T]}(X_t^{n,m} \wedge 1)] \downarrow 0.$$

Otherwise, there exists $\varepsilon > 0$ such that $\hat{E}[\sup_{m>n}\sup_{t\in[0,T]}(X_t^{n,m} \wedge 1)] > \varepsilon$ for all $n \in N$. Consequently, for each $n \in N$, there exists $m(n) > n$ such that $\hat{E}[\sup_{t\in[0,T]}(X_t^{n,m(n)} \wedge 1)] > \varepsilon$.

Noting that

$$\varepsilon < \hat{E}[\sup_{t\in[0,T]}(X_t^{n,m(n)} \wedge 1)] \leq c(\sup_{t\in[0,T]} X_t^{n,m(n)} > \varepsilon/2) + \varepsilon/2,$$

we have $c(\sup_{t\in[0,T]} X_t^{n,m(n)} > \varepsilon/2) > \varepsilon/2$. Since

$$[\sup_{t\in[0,T]} X_t^{n,m(n)} > \varepsilon/2] = \pi(\{(\omega,t)|\ X_t^{n,m(n)}(\omega) > \varepsilon/2\}),$$

the projection of $\{(\omega,t)|\ X_t^{n,m(n)}(\omega) > \varepsilon/2\}$ on Ω, we have stopping time $\tau_n \leq T$ such that $\hat{E}(X_{\tau_n}^{n,m(n)}) > \varepsilon^2/4$ by section theorem.

On the other hand, by Theorem 4.5, $\{X_t^{n,m(n)}\}$ has the following decomposition

$$X_t^{n,m(n)} = M_t^n - K_t^n,$$

where $\{M_t^n\}$ is a symmetric G-martingale and $\{-K_t^n\}$ a negative G-martingale with $M_T^n, K_T^n \in L_G^1(\Omega_T)$. By Theorem 4.5 and Corollary 5.2, $\{M_t^n\}, \{-K_t^n\}$ can be taken to be quasi-continuous.

So we have as $n \to \infty$

$$\begin{aligned} & \hat{E}(X_{\tau_n}^{n,m(n)}) \\ \leq\ & \hat{E}(M_{\tau_n}^n) + \hat{E}(-K_{\tau_n}^n) \\ \leq\ & \hat{E}(M_T^n) \\ \leq\ & \hat{E}(X_T^{n,m(n)}) + \hat{E}(K_T^n) \\ \leq\ & 2\hat{E}(X_T^{n,m(n)}) \to 0. \end{aligned}$$

This is a contradiction.

Therefore, by the same arguments as in Corollary 5.2, $\{M_t\}$ has a quasi-continuous version. $\square$

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